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Dynamics of texture for spinor condensates

Simple analytic solutions

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Abstract. We study a dynamics of texture for a two-component spinor bose condensate. This is carried out by adopting a time dependent Landau-Ginzburg Lagrangian for a spinor order parameter. By using a polar form of the spinor order parameter, we obtain a field equation for the texture. In particular we consider a one dimensional model in which we can obtain analytic forms for the textures in terms of elliptic functions of several kinds. We find that these solutions are characterized by a modulus parameter, and changes in this parameter cause structural changes of texture.

PACS. 03.75.Mn Multicomponent condensates; Spinor condensates

1 Introduction

One of the most interesting subjects in condensed matter physics is the study of the behavior of order parameters. This may be done using a Landau-Ginzburg theory that describes both the thermodynamic features and spatial pattern of condensed states of materials [1]. The simplest form of order parameter is given by a scalar function of a real or complex variable. Typical examples are superfluid He4 and superconductivity [1,2], in which the order parameters are given by complex wave functions (macrowave functions). On the other hand, it is well known that there are many condensed systems which can be described by an order parameter with several components. One of these has been realized with the discovery of multicomponent bose condensates, which have been formed by the simultaneous trapping and cooling of atoms in distinct hyperfine or spin levels [3–6]. The theoretical analysis of spinor bose condensates has been developed for the case of hyperfine spin 1 states (see e.g. [7,8]). The theoretical analysis of the Bose-Einstein condensates has been reviewed for example by Legget [9].

Condensed states described by order parameters with several components can be characterized as anisotropic fluids. Superfluid He3A is regarded as a typical example of a quantum anisotropic fluid [10] and a liquid crystal is a classical anisotropic fluid [11]. The characteristic concept in anisotropic fluids is structure or *texture*. Intuitively speaking, texture is a spatial pattern described by a *directional field* inherent in the components of the order parameter. The typical examples are the director vector in a liquid crystal and the l-vector in superfluid He3A.

Having given a brief description of the features of anisotropic quantum fluids, we address the dynamics of texture in a spinor bose condensate. Specifically, we consider the case that the order parameter has two components. In order to treat this problem, we adopt a time-dependent Landau-Ginzburg (LG) Lagrangian, which was previously used by one of the authors [12]. As a special case, we consider a one dimensional model, which leads to analytic forms for the texture in terms of elliptic functions. The LG Lagrangian adopted here is an extension of that used for a scalar order parameter in superfluid He4 [2,13] or for a vector order parameter in superfluid He3A [14].

2 General formulation

2.1 Lagrangian for the spinor bose field

To study a two component spinor Bose-Einstein Condensate (BEC), we introduce the following order parameter.

$$\Psi(\boldsymbol{r},t) = \begin{pmatrix} \psi_1(\boldsymbol{r},t) \\ \psi_2(\boldsymbol{r},t) \end{pmatrix}. \tag{1}$$

The initial time dependent LG Lagrangian is

$$\mathcal{L} = \frac{1}{2}\hbar i\left(\Psi^\dagger\frac{\partial\Psi}{\partial t} - \frac{\partial\Psi^\dagger}{\partial t}\Psi\right) - \mathcal{H}[\Psi,\Psi^\dagger] \tag{2}$$

$$\mathcal{H}[\Psi,\Psi^\dagger] = \left[\frac{\hbar^2}{2m}(\nabla\Psi^\dagger)(\nabla\Psi) - \mu\Psi^\dagger\Psi + U[\Psi,\Psi^\dagger]\right] \tag{3}$$

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where $H[\Psi,\Psi^\dagger]$ is the Hamiltonian of the bose components[1]. The last term in the Hamiltonian represents the interactions between constituent particles, which is given by quartic functions of the order parameter. From the various choices of quartic interaction, we consider special cases in which the interaction is given in terms of products of bilinear forms, $(\Psi^\dagger\sigma_j\Psi)(\Psi^\dagger\sigma_k\Psi)$ or $(\Psi^\dagger\Psi)^2$, where, σ_j is a Pauli matrix. In particular we choose

$$U[\Psi,\Psi^\dagger] = g_1(\Psi^\dagger\Psi)^2 - g_2(\Psi^\dagger\sigma_z\Psi)^2. \tag{4}$$

By using the variational principle, we obtain a Gross-Pitaevskii type equation of motion for ψ_1, ψ_2.

$$i\hbar\frac{\partial\psi_1}{\partial t} = -\left(\frac{\hbar^2}{2m}\nabla^2+\mu\right)\psi_1 + g|\psi_1|^2\psi_1 + A|\psi_2|^2\psi_1$$
$$i\hbar\frac{\partial\psi_2}{\partial t} = -\left(\frac{\hbar^2}{2m}\nabla^2+\mu\right)\psi_2 + g|\psi_2|^2\psi_2 + A|\psi_1|^2\psi_2. \tag{5}$$

Here, $A = 2(g_1+g_2)$ and $g = 2(g_1-g_2)$. The set of equations (5) shows that the number of each component is conserved.

2.2 Reduction to the hydrodynamical form

We rewrite the two complex fields ψ_1, ψ_2 in an alternative form. Two complex scalar fields mean that there are four degrees of freedom, which will be expressed by four real functions $n(\boldsymbol{r},t),\theta(\boldsymbol{r},t),\varphi(\boldsymbol{r},t),\alpha(\boldsymbol{r},t)$. By using these real functions, the order parameter can be written in the form

$$\Psi = \sqrt{n}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{-i\varphi}\sin\frac{\theta}{2}\end{pmatrix}e^{-i\alpha} \tag{6}$$

where $\sqrt{n(\boldsymbol{r},t)}$ is the amplitude which represents the total density of two kinds of particle and $\theta(\boldsymbol{r},t),\varphi(\boldsymbol{r},t),\alpha(\boldsymbol{r},t)$ are the phases. Also, we define the spin field as

$$S_k = \frac{1}{n}\Psi^\dagger\sigma_k\Psi \tag{7}$$

where $k=x,y,z$, and σ_k is the Pauli matrix

$$\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}\quad \sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\quad \sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{8}$$

Using these definitions, $\boldsymbol{S}=(S_x,S_y,S_z)$ can be written as

$$S_x = \sin\theta\cos\varphi,\ S_y=\sin\theta\sin\varphi,\ S_z=\cos\theta. \tag{9}$$

Thus, we can express the Lagrangian in terms of the new variables.

$$\mathcal{L} = \hbar n\left(\frac{\partial\alpha}{\partial t}+\frac{\partial\varphi}{\partial t}\sin^2\frac{\theta}{2}\right) - \mathcal{H} \tag{10}$$
$$\mathcal{H} = \mathcal{H}_{\rm kin} + \mathcal{H}_{\rm int}. \tag{11}$$

[1] In the following arguments, we omit the term coming from the confined potential, because this does not play an important role for describing the motion of texture.

Here, the first term in the Lagrangian (10), which includes the first order time derivatives of φ and α, determines the dynamical behavior of bose condensates. On the other hand, the second term represents the kinetic energy and interaction energy, which are given by

$$\begin{aligned}\mathcal{H}_{\rm kin} = \frac{\hbar^2}{2m}\Big\{&(\nabla\sqrt{n})^2+\frac{1}{4}n(\nabla\theta)^2\\ &+2n(\nabla\varphi)\cdot(\nabla\alpha)\sin^2\frac{\theta}{2}\\ &+n(\nabla\varphi)^2\sin^2\frac{\theta}{2}+n(\nabla\alpha)^2\Big\}\end{aligned} \tag{12}$$

$$\begin{aligned}\mathcal{H}_{\rm int} = -\mu n &+ \frac{1}{2}gn^2\left(\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}\right)\\ &+An^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}.\end{aligned} \tag{13}$$

It should be noted that $\mathcal{H}_{\rm int}$ does note include φ and α.

Now, we examine the content of kinetic terms of Hamiltonian. This can be divided into two terms.

$$\mathcal{H}_{\rm kin1} = \frac{\hbar^2}{2m}n\left\{(\nabla\alpha)+(\nabla\varphi)\sin^2\frac{\theta}{2}\right\}^2 \tag{14}$$

$$\mathcal{H}_{\rm kin2} = \frac{\hbar^2}{2m}\left\{(\nabla\sqrt{n})^2+\frac{1}{4}n(\nabla\theta)^2+\frac{1}{4}n(\nabla\varphi)^2\sin^2\theta\right\}. \tag{15}$$

Equation (14) gives the fluid kinetic energy for the anisotropic fluid, whereas equation (15) gives the internal energy. In fact the velocity field of the anisotropic fluid is defined by

$$\boldsymbol{v}_{\rm s} = \frac{\hbar}{m}\left\{(\nabla\alpha)+(\nabla\varphi)\sin^2\frac{\theta}{2}\right\} \tag{16}$$

which is also derived from the current

$$\boldsymbol{j} = \frac{\hbar}{2i}\left\{\Psi^\dagger(\nabla\Psi)-(\nabla\Psi^\dagger)\Psi\right\} = mn\boldsymbol{v}_s. \tag{17}$$

Thus, we can write $\mathcal{H}_{\rm kin1}$ as

$$\mathcal{H}_{\rm kin1} = \frac{1}{2}mn\boldsymbol{v}_{\rm s}^2 = \frac{m}{2n}\boldsymbol{j}^2. \tag{18}$$

On the other hand, $\mathcal{H}_{\rm kin2}$ consists of two terms: one is the term given by the density gradient and the other is the term given by the gradient of each component of spin. Thus we have

$$\mathcal{H}_{\rm kin2} = \frac{\hbar^2 n}{2m}\left\{(\nabla\log n)^2+\frac{1}{4}(\nabla\boldsymbol{S})^2\right\} \tag{19}$$

where $\nabla\boldsymbol{S} = (\nabla S_x,\nabla S_y,\nabla S_z)$. The latter term is similar to the continuous limit of the Heisenberg model [15]. Alternatively, this corresponds to the *bending energy* of elasticity theory [16].

2.3 Equation of motion for the texture

We shall restrict the general argument given above by imposing the following conditions. (i) The density $n(\boldsymbol{r},t)$ is regarded as constant. This situation corresponds to the London limit in superfluid He3. That is, the magnitude of the order parameter is constant and the only angular components $(\theta(\boldsymbol{r},t),\varphi(\boldsymbol{r},t))$ are allowed to vary. An analogous mechanical example would be a spherical pendulum for which the length of the pendulum is fixed. (ii) The overall phase is taken as $\alpha(\boldsymbol{r},t)=0$, which corresponds to gauge fixing. Thus the Lagrangian becomes simply

$$\mathcal{L} = \hbar n \frac{\partial \varphi}{\partial t}\sin^2\frac{\theta}{2} - \mathcal{H} \tag{20}$$

$$\begin{aligned}\mathcal{H} = \frac{\hbar^2 n}{2m}&\left\{\frac{1}{4}(\nabla\theta)^2 + (\nabla\varphi)^2\sin^2\frac{\theta}{2}\right\}\\ &-\mu n + \frac{1}{2}g n^2\left(\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}\right) + A n^2 \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}.\end{aligned} \tag{21}$$

From the variational principle $\delta\int dt d^3\boldsymbol{x}\mathcal{L}=0$, we obtain the equations of motion

$$\dot{\theta} = -\frac{2}{\hbar n}\frac{1}{\sin\theta}\frac{\delta\mathcal{H}}{\delta\varphi} \quad \dot{\varphi} = \frac{2}{\hbar n}\frac{1}{\sin\theta}\frac{\delta\mathcal{H}}{\delta\theta} \tag{22}$$

where $\frac{\delta}{\delta f} = \frac{\partial}{\partial f} - \nabla\cdot\frac{\partial}{\partial(\nabla f)}$ means the functional derivative. Using the concrete form of the Hamiltonian, we obtain

$$\begin{aligned}\frac{\partial\theta}{\partial t} &= \frac{\hbar}{m}\frac{1}{\sin\theta}\nabla\cdot\Big[(\nabla\varphi)(1-\cos\theta)\Big]\\ \frac{\partial\varphi}{\partial t} &= \frac{\hbar}{2m}\left[(\nabla\varphi)^2 - \frac{1}{\sin\theta}\nabla^2\theta\right] + \frac{4g_2 n}{\hbar}\cos\theta.\end{aligned} \tag{23}$$

These equations are the starting point for the following discussion.

3 Analytic solvable model: one dimensional case

3.1 Solving the problem

Now we examine the behavior of texture by studying specific models. For this purpose, we restrict our discussion to the one dimensional case. Although a model with one spatial dimension seems to be rather artificial, the possibility of experimental realization may be not excluded. Such a situation may be realized by a BEC in thin tube. For example we consider the case that the BEC is trapped in y,z directions by potentials $V(y),V(z)$, whereas there is no potential in the x direction. We choose the order parameter $\Psi = {}^t\big(\Phi_1(x,t),\Phi_2(x,t)\big)\times Y(y)Z(z)$ where $Y(y),Z(z)$ are normalized wave packets, satisfying the steady state Schrödinger equation. Then by averaging the action functional over these wave packets, we have the effective coupling constants g', A' (or g_1', g_2') and chemical potential μ'. The details of this procedure are given in the appendix. Hereafter we will write the effective coupling constant g_2' simply as g_2. Furthermore, we choose the coupling constant g_2 to be positive. In the following argument, we adopt several *ansatze* for the angular fields θ and φ, which enable us to obtain explicit analytic solutions in terms of elliptic functions.

Ansatz 1: $\theta(x,t)=\theta(x)$, $\varphi(x,t)=$ const. — In this case, the equation of motion becomes

$$\frac{d^2\theta}{dx^2} = \frac{8mng_2}{\hbar^2}\sin\theta\cos\theta \tag{24}$$

which is identical with that of pendulum. Hence, it can be integrated to give

$$\left(\frac{d\theta}{dx}\right)^2 = \frac{mn}{\hbar^2}\left\{8g_2\sin^2\theta + (C-4g_2)\right\} \tag{25}$$

where C is determined by the initial conditions θ_0 and $d\theta_0/dx$. The solution of equation (24) is given in terms of Jacobi elliptic functions, which can be classified into three cases, according to the value of the modulus parameter defined by

$$\lambda = \sqrt{\frac{8g_2}{4g_2+C}}. \tag{26}$$

(a) $\lambda<1$: this corresponds to the rotating pendulum. The solution of this case is

$$\cos\theta = \mathrm{sn}\big(\gamma_1(x+k)\ \big|\ \lambda\big) \tag{27}$$

where γ_1 and k are defined by

$$\begin{aligned}\gamma_1 &= \sqrt{\frac{mn(4g_2+C)}{\hbar^2}}\\ k &= \frac{1}{\gamma_1}\mathrm{F}\left(\theta_0+\frac{\pi}{2}\ \Big|\ \lambda\right).\end{aligned} \tag{28}$$

(b) $\lambda>1$: this corresponds to the oscillating pendulum. The range of oscillation is $-\lambda^{-1}\le\cos\theta\le\lambda^{-1}$. The solution is given by

$$\cos\theta = -\lambda^{-1}\mathrm{sn}\big(\gamma_2(x+k)\ \big|\ \lambda^{-1}\big) \tag{29}$$

where γ_2 and k are defined by

$$\begin{aligned}\gamma_2 &= \sqrt{\frac{8mng_2}{\hbar^2}}\\ k &= -\frac{1}{\gamma_2}\mathrm{F}\left(\arcsin\big(\lambda\cos\theta_0\big)\ \Big|\ \lambda^{-1}\right).\end{aligned} \tag{30}$$

(c) $\lambda=1$: in this critical case, there are two solutions. One is

$$\cos\theta = \frac{1-k^2\exp[2\gamma_2 x]}{1+k^2\exp[2\gamma_2 x]} \tag{31}$$

where $k=\tan\frac{\theta_0}{2}$. This solution has the property that $\cos\theta\to-1$ for $x\to\infty$ The other is

$$\cos\theta = \frac{1-k^2\exp[-2\gamma_2 x]}{1+k^2\exp[-2\gamma_2 x]} \tag{32}$$

which behaves as $\cos\theta\to1$ for $x\to\infty$. These solutions exhibit a kink-like behavior.

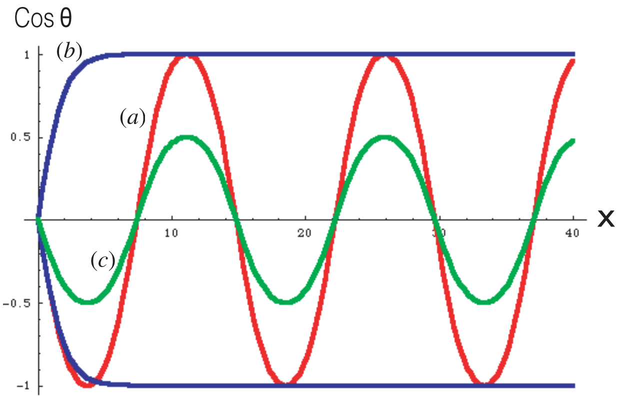

Fig. 1. Profile of $\cos\theta$ for the case of ansatz 1. The labels (a), (b), (c) correspond to the cases indicated in the text.

The behaviors of the three solutions are depicted in Figure 1.

Ansatz 2: $\theta(x,t) = \theta(x)$, $\varphi(x,t) = \varphi(x)$ — This is a general case of *ansatz* 1. The equation of motion becomes

$$\frac{d^2\theta}{dx^2} = 2\left(\frac{d\varphi}{dx}\right)^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2} + \frac{8mng_2}{\hbar^2}\sin\theta\cos\theta$$
$$\frac{\hbar^2}{m}\frac{d}{dx}\left(\frac{d\varphi}{dx}\sin^2\frac{\theta}{2}\right) = 0. \tag{33}$$

By using the integral derived from the second equation, we have a reduced equation for θ

$$\left(\frac{d\theta}{dx}\right)^2 = \frac{m}{\hbar^2}\frac{1}{1-\cos\theta}\Big[-C^2 + D(1-\cos\theta) - 4ng_2(2\cos^2\theta - 1)(1-\cos\theta)\Big] \tag{34}$$

where C, D are determined by the initial conditions, θ_0, φ_0, $d\varphi_0/dx$ and $d\theta_0/dx$. It can be rewritten as

$$\left(\frac{dx}{d\cos\theta}\right)^2 = \frac{\hbar^2}{8mng_2}\frac{1}{F(\cos\theta)} \tag{35}$$

where $F(\cos\theta)$ is a quartic function of $\cos\theta$. We have two cases according to the choice of initial conditions: (i) $F(\cos\theta)$ has four real roots $[-1,\alpha,\beta,\gamma]$ (such that $-1 < \alpha < \beta < \gamma$) thus

$$F(\cos\theta) = (\cos\theta+1)(\cos\theta-\alpha)(\cos\theta-\beta)(\cos\theta-\gamma) \tag{36}$$

and (ii) $F(\cos\theta)$ has two real roots and two complex roots which are conjugate to each other. In what follows we restrict our consideration to case (i) for which the solution is given by

$$\cos\theta = \frac{\alpha(\beta+1) + (\beta-\alpha)\mathrm{sn}^2\big(\Gamma(x+k)\,\big|\,\Lambda\big)}{(\beta+1) - (\beta-\alpha)\mathrm{sn}^2\big(\Gamma(x+k)\,\big|\,\Lambda\big)}$$
$$\varphi = \frac{\sqrt{2m}}{\hbar}C\int\frac{dx}{1-\cos\theta} \tag{37}$$

where

$$k = \frac{1}{\Gamma}\mathrm{F}\left(\arcsin\left(\sqrt{\frac{(\beta+1)(\cos\theta_0-\alpha)}{(\beta-\alpha)(\cos\theta_0+1)}}\right)\middle|\,\Lambda\right)$$
$$\Gamma = \frac{\sqrt{8mng_2(\gamma-\alpha)(\beta+1)}}{2\hbar}$$
$$\Lambda = \sqrt{\frac{(\beta-\alpha)(\gamma-1)}{(\beta+1)(\gamma-\alpha)}}. \tag{38}$$

This corresponds to an oscillating pendulum, and the behavior of the solution is shown in Figure 2.

Ansatz 3: $\theta(x,t) = \theta(x)$, $\varphi(x,t) = \omega t/4$ — This is a model in which φ rotates with constant angular velocity. The equation of motion is thus

$$\frac{d^2\theta}{dx^2} = \frac{m\omega}{2\hbar}\sin\theta + \frac{8mng_2}{\hbar^2}\sin\theta\cos\theta \tag{39}$$

which can be integrated to give

$$\left(\frac{d\theta}{dx}\right)^2 = \frac{m}{\hbar^2}\Big[-\hbar\omega\cos\theta - 4g_2n\cos 2\theta + C\Big] \tag{40}$$

where C is determined by the initial conditions θ_0 and $d\theta_0/dx$. Equation (39) can be rewritten as

$$\left(\frac{dx}{d\cos\theta}\right)^2 = \frac{\hbar^2}{8mng_2}\frac{1}{(\cos\theta+1)(\cos\theta-c_-)} \times\frac{1}{(\cos\theta-c_+)(\cos\theta-1)} \tag{41}$$

where $c_- \le c_+$. Thus, the solutions of equation (40) are given by elliptic integrals, which can be classified into three cases according to the values of c_+ and c_-: (a') $c_- \le -1 \le \cos\theta \le 1 \le c_+$ (b') $c_- \le -1 \le \cos\theta \le c_+ \le 1$ (c') $-1 \le c_- \le \cos\theta \le c_+ \le 1$.

(a') For the case $c_- \le -1 \le \cos\theta \le 1 \le c_+$,

we have

$$\cos\theta = \frac{2c_-\mathrm{sn}^2\big(\Gamma_1(x+k)\,\big|\,\Lambda\big) + (1-c_-)}{2\mathrm{sn}^2\big(\Gamma_1(x+k)\,\big|\,\Lambda\big) - (1-c_-)} \tag{42}$$

where

$$k = \frac{1}{\Gamma_1}\mathrm{F}\left(\arcsin\left(\sqrt{\frac{(1-c_-)(\cos\theta_0+1)}{2(\cos\theta_0-c_-)}}\right)\middle|\,\Lambda\right)$$
$$\Gamma_1 = \sqrt{\frac{2mng_2(1+c_+)(1-c_-)}{\hbar^2}}$$
$$\Gamma_2 = \sqrt{\frac{4mng_2(c_+-c_-)}{\hbar^2}}$$
$$\Lambda = \sqrt{\frac{2(c_+-c_-)}{(1+c_+)(1-c_-)}}. \tag{43}$$

This corresponds to the oscillating pendulum.

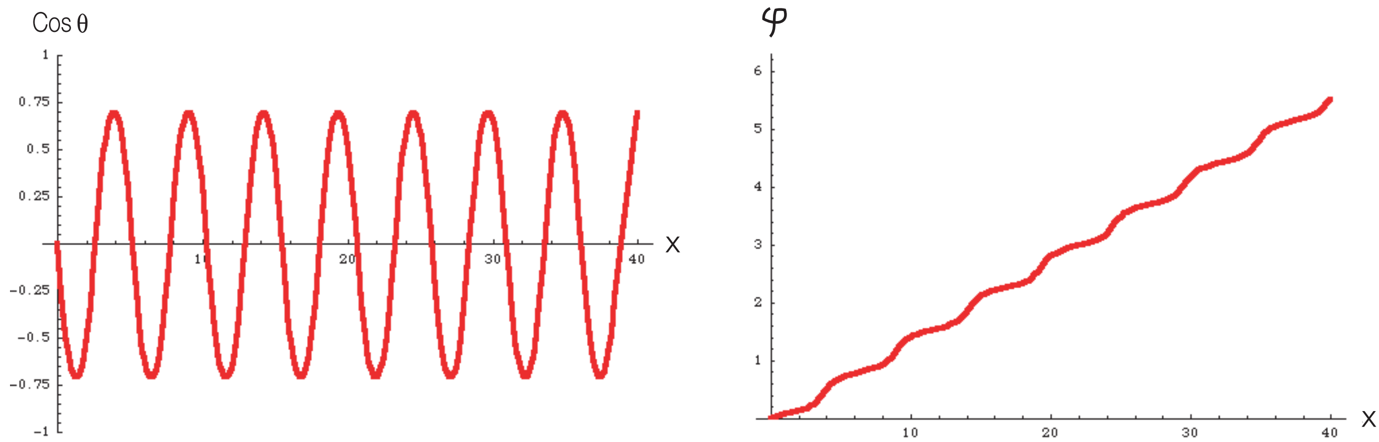

Fig. 2. Profile of $\cos\theta$ and φ for the case of ansatz 2.

(b′) For the case $c_- \leq -1 \leq \cos\theta \leq c_+ \leq 1$,

$\cos\theta$ oscillates in the range $[-1, c_+]$, such that

$$\cos\theta = \frac{c_-(1+c_+)\mathrm{sn}^2\big(\Gamma_2(x+k)\,\big|\,\Lambda^{-1}\big)+(c_+-c_-)}{(1+c_+)\mathrm{sn}^2\big(\Gamma_2(x+k)\,\big|\,\Lambda^{-1}\big)-(c_+-c_-)}$$

$$k = \frac{1}{\Gamma_2}\mathrm{F}\left(\arcsin\left(\sqrt{\frac{(c_+-c_-)(\cos\theta_0+1)}{(1+c_+)(\cos\theta_0-c_-)}}\right)\,\middle|\,\Lambda^{-1}\right). \tag{44}$$

(c′) For the case $-1 \leq c_- \leq \cos\theta \leq c_+ \leq 1$,

$\cos\theta$ oscillates in the range $[c_-, c_+]$, such that

$$\cos\theta = \frac{c_-(c_++1)+(c_+-c_-)\mathrm{sn}^2\big(\Gamma_1(x+k)\,\big|\,\Lambda\big)}{(c_++1)-(c_+-c_-)\mathrm{sn}^2\big(\Gamma_1(x+k)\,\big|\,\Lambda\big)}$$

$$k = \frac{1}{\Gamma_1}\mathrm{F}\left(\arcsin\left(\sqrt{\frac{(c_++1)(\cos\theta_0-c_-)}{(c_+-c_-)(\cos\theta_0+1)}}\right)\,\middle|\,\Lambda\right). \tag{45}$$

The behaviors of these solutions are shown in Figure 3.

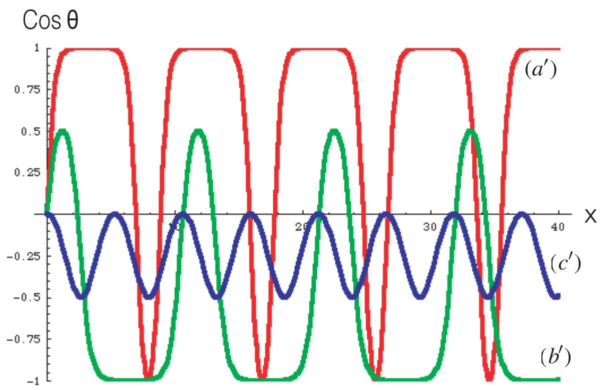

Fig. 3. Profile of $\cos\theta$ for the case of ansatz 3. The labels (a′), (b′), (c′) corresponds to the cases indicated in the text.

3.2 Remarks

The solvable models considered here have the peculiar feature that the solutions can be written in terms of elliptic functions which are controlled by specific parameters that are inherent in the solutions themselves. These parameters, λ, Λ, are called *modulus parameters* and are determined by the initial conditions as well as the coupling constant. A structural change of texture occurs as the modulus parameter varies. This feature may be regarded as a "cross over" between solutions, and this may be considered as a characteristic of the one dimensional model.

4 Brief summary

We have studied the dynamics of texture inherent in spinor bose condensates, specifically for the case of a two-component bose condensate. This has been formulated by using time-dependent Landau-Ginzburg theory. The analytic form of the texture has been written down in terms of elliptic functions for a one dimensional model by adopting specific for the form of the texture.

There are two obvious problems following on from this work. One is to extend the present idea to 2 or 3 dimensions, which may reveal more characteristic aspects of texture, and in particular, the dynamics of a vortex. The other is to extend the argument from a two component spinor to the case of a general spinor. This has been briefly discussed by using the SU(2) coherent state [12] and further studies need to be carried out. These topics are left for forthcoming papers.

We would like to thank Professor H. Yabu for useful suggestions.

Appendix

Here we give the details of the derivation of the quasi-one dimensional effective action that is used in Section 3. The action functional which includes the confining potential in the y and z directions is

$$S[\Psi,\Psi^\dagger] = \int dt \int dxdydz \left[\frac{i\hbar}{2}\left(\Psi^\dagger\frac{\partial\Psi}{\partial t} - \frac{\partial\Psi^\dagger}{\partial t}\Psi\right) + \frac{\hbar^2}{2m}(\nabla\Psi^\dagger)(\nabla\Psi) - (\mu - V_y(y) - V_z(z))\Psi^\dagger\Psi + U[\Psi,\Psi^\dagger]\right]. \quad (46)$$

Let $Y(y)$ satisfy the steady state Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2Y(y)}{dy^2} + V(y)Y(y) = \epsilon_y Y(y) \quad (47)$$

with a similar equation for $Z(z)$. Then we obtain the effective action for Φ_1 and Φ_2

$$S_{\text{eff}} = \int dt \int dx \left[\sum_{i=1,2}\left\{\frac{i\hbar}{2}\left(\Phi_i^*\frac{\partial\Phi_i}{\partial t} - \frac{\partial\Phi^*}{\partial t}\Phi_i\right) + \frac{\hbar^2}{2m}\left|\frac{\partial\Phi_i}{\partial x}\right|^2 - \mu'|\Phi_i|^2 + g'|\Phi_i|^4\right\} + A'|\Phi_1|^2|\Phi_2|^2\right] \quad (48)$$

where we define the new parameters

$$\mu' = \mu + \epsilon_y + \epsilon_z, \quad g' = g\int dydz|Y(y)|^4|Z(z)|^4$$
$$A' = A\int dydz|Y(y)|^4|Z(z)|^4. \quad (49)$$

According to the above calculation, we can see that the trapping of the BEC in two directions changes the coupling constants. For example, when we choose the potential and wave packets as

$$V(y) = \frac{1}{2}m\omega^2y^2, \quad Y(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left(-\frac{m\omega}{2\hbar}y^2\right) \quad (50)$$

with a similar expression for $Z(z)$, the coupling constants become

$$\mu' = \mu + \hbar\omega, \quad g' = \frac{m\omega g}{8\hbar}, \quad A' = \frac{m\omega A}{2\hbar}. \quad (51)$$

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